ZEROES OF L-SERIES IN CHARACTERISTIC p

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Abstract. In the classical theory of L-series, the exact order (of zero) at a trivial zero is easily computed via the functional equation. In the characteristic p theory, it has long been known that a functional equation of classical $s \mapsto 1-s$ type could not exist. In fact, there exist trivial zeroes whose order of zero is "too high;" we call such trivial zeroes "non-classical." This class of trivial zeroes was originally studied by Dinesh Thakur [Th2] and quite recently, Javier Diaz-Vargas [DV2]. In the examples computed it was found that these non-classical trivial zeroes were correlated with integers having *bounded* sum of p-adic coefficients. In this paper we present a general conjecture along these lines and explain how this conjecture fits in with previous work on the zeroes of such characteristic p functions. In particular, a solution to this conjecture might entail finding the "correct" functional equations in finite characteristic.

1. Introduction

The debt all mathematicians owe to Euler is obvious and universally known. Euler's instincts and mathematical taste have had the most profound effect on all subsequent generations of researchers. Nowhere is this more evident than in Euler's fantastic contributions to number theory and, in his work on number theory (see, for instance, [Du1]), nothing is more fabulous than Euler's investigation into what we now call the Riemann zeta function $\zeta(s) = \zeta_{\mathbb{Q}}(s) := \sum_{n=1}^{\infty} n^{-s}$. In fact, Euler was the first to appreciate that $\zeta(s)$ had a functional equation, (see [Ay1] for a wonderful discussion of Euler's insights).

The fact that there are infinitely many primes goes back to Euclid. Euler gave an elegant refinement of this result by establishing that

$$\sum_{p \text{ prime}} \frac{1}{p}$$

diverges thereby giving some indication of how well spaced the primes actually are. Euler's proof of this fact uses the "Euler product" associated to $\zeta(s)$ as well as the divergence of the harmonic series $\zeta(1) = \sum_{n=1}^{\infty} \frac{1}{n}$.

Euler was also fascinated by the special values of $\zeta(s)$ at all the other integers, both positive and negative, and he devoted much energy to their computation. In fact, it was by studying these values that Euler discovered the above mentioned functional equation. A modern number theorist cannot read these discoveries of Euler without wonder at the sheer beauty and audacity of Euler's methods. While the rigorous deduction of the functional equation of $\zeta(s)$ had to wait until Riemann and the methods of complex analysis, Euler's argument makes thrilling use of both convergent *and* divergent series (see Section 2 below).

In the process of evaluating $\zeta(s)$ where s is an integer, Euler discovered the "trivial zeroes" of $\zeta(s)$ at the negative even integers. These zeroes play a crucial role in Euler's calculations

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This work is in honor of my mother Barbara Goss Alter.

as they form the numerator of a rational quantity Euler needs to compute *and* where his methods (and those of everybody else since) fail to compute the denominator in closed form (see Remark 1 and Equation 6). Moreover, the functional equation of $\zeta(s)$ then allows one to then show that these trivial zeroes are simple. As is well-known in modern number theory, the functional equation of $\zeta(s)$ is merely the beginning of a vast undertaking whose goal is to show that *all* arithmetically interesting Dirichlet series ultimately behave in similar ways.

Euler's work on $\zeta(s)$ has always been an inspiration in our work on L-series in characteristic p. We will briefly review the definition of such characteristic p functions in Section 3 where the base ring A of the theory of Drinfeld modules plays the role of the integers $\mathbb{Z}$ in classical theory. In particular, in the prototypical case of $A = \mathbb{F}_r[T]$, $r = p^m$, p prime, (which, like $\mathbb{Z}$, is also Euclidean) one is able to describe a function $\zeta_A(s) := \sum_{n \in A \text{ monic}} n^{-s}$ (see Chapter 8 of [Go2]) which is a very close cousin of $\zeta(s)$. Indeed, using the period of the Carlitz module instead of $2\pi i$, one could readily establish an analog of Euler's calculation of $\zeta(2n)$ $n = 1, 2, \cdots$ for those positive j which are "A-even" (i.e. j divisible by $r - 1 = \#A^*$). At the negative integers, $-j$ ($j \geq 0$) one obtains divergent sums of the form $\sum_{n \in A \text{ monic}} n^j$ which, upon regrouping according to the degree of n, become finite. When j is again A-even, this sum is 0 giving a "trivial zero."

Thus, on the surface, the special values of $\zeta_A(s)$ behave very similarly to those of $\zeta_{\mathbb{Q}}(s)$, and so a first hope would be to follow Euler and guess at "the functional equation" for $\zeta_{\mathbb{F}_r[T]}(s)$. This fails to work for two basic reasons: 1. Obviously the mapping $s \mapsto 1 - s$ is a bijection between even and odd integers; this fails for A-odd and A-even numbers when $r \neq 3$; 2. Even with $r = 3$ there are *two* distinct analogs of Bernoulli numbers in the characteristic p theory; this results in considerably more complicated quotients than in classical theory.

This state of affairs persisted until the mid 1990's when two seemingly independent developments occurred. In the first [Wa1] (see also [DV1]) Daqing Wan computed the Newton polygons associated to $\zeta_A(s)$ when $r = p$ (later extended to all r by B. Poonen and J. Sheats [Sh1]) thereby establishing that the absolute value of a zero *uniquely* determines it (including the multiplicity of the zero). Therefore the zeroes of $\zeta_{\mathbb{F}_r[T]}$ lie "on the line" $\mathbb{F}_r((1/T))$ and are simple. Wan's calculations were prompted by the observation by the present author that, in some cases at least, the coefficients of $\zeta_A(s)$ go to 0 exponentially. Such a rate of decay is far faster than is necessitated to simply establish the basic analyticity properties of such a function and is implied by having the degrees of certain "special polynomials" (see Section 3) grow logarithmically. Such logarithmic growth is now known to be a completely general phenomenon [Boc1], [Go5].

In the second development, D. Thakur [Th2] looked into the possibility that, for general A, the trivial zeroes had higher order multiplicities; such a phenomenon *never* happens classically. In other words, the construction of trivial zeroes comes equipped with a "classical" lower-bound on the order of zero. However as Thakur found, there are many instances where this lower-bound is *not* the exact order; such a trivial zero is called "non-classical." It is totally remarkable, and very important for us, that Thakur's results on non-classical trivial zeroes involve having the sum of the p-adic digits of the trivial zero be *bounded.* Thakur's computations have recently been extended by Javier Diaz-Vargas ([DV2]). These basic results will be recalled in Section 4.

The results of Wan, Sheats, etc., are clearly a type of "Riemann hypothesis" (see [Go3], [Go4]) and one wants to be able to put them into a general conjecture about the zeroes

in *full* generality for all motives (τ-sheaves, etc.) and interpolations at all places of the quotient field k of A. Our first attempt to do so [Go3] imply ignored the trivial zeroes (as they are ignored classically in the Riemann hypothesis). As was reported in [Go4], this conjecture was wrong *precisely* because of the impact of higher order trivial zeroes! More precisely, using the topology on the domain space of our L-series, one is able to use higher order trivial zeroes to inductively construct p-adic integers where the conjecture is false. The construction produces such p-adic integers by building up their canonical p-adic expansion out of the expansions of trivial zeroes with higher orders.

It is precisely here that the computations of Thakur and Diaz-Vargas now fit. Indeed, their computations lead naturally to the conjecture (Conjecture 1) that those integers j for which the trivial zero at $-j$ is non-classical have *bounded* sum of their p-adic digits. Presenting this conjecture is the goal of this work and we discuss the conjecture both at ∞ (i.e., the analog of the complex field) *and* at the interpolations of our functions at finite primes. If this conjecture is true, then it places a limit on how one can construct counter-examples to our original conjecture. Indeed, it implies that we need not worry about non-classical trivial zeroes *alone* leading to counter-examples as they can have no effect on our construction once the sum of the p-adic digits of the integers being used becomes sufficiently large (see the discussion in Section 5; in particular, Example 4). We view this as positive evidence for the conjecture.

As the reader may see, the trivial zeroes play a special role in both the classical and characteristic p theory. But what is the right general conjecture on the zeroes in characteristic p? As of now, we do not know. However, since the functional equation classically is what allows one to compute the order of a trivial zero, it seems to us quite reasonable that a proof of the above conjecture in our case will generate the correct ideas and techniques.

It is clear that this work owes a great deal to Euler. It should also be clear that it owes a great deal to the mathematical taste and insight of Dinesh Thakur and Javier Diaz-Vargas. It is moreover my pleasure to thank Thakur and J.-P. Serre for helpful comments.

2. Euler's discovery of the functional equation for $\zeta_{\mathbb{Q}}(s)$

Our treatment here follows that of [Ay1]; the reader is referred there for references and any elided details. Let $\zeta(s)$ be the Riemann zeta function.

After many years of work, Euler succeeded in computing the values $\zeta(2n)$, $n = 1, 2 \cdots$ in terms of Bernoulli numbers. Euler then turned his attention to the values $\zeta(s)$ at negative integers. Of course, Euler did not have analytic continuation to work with and relied on his instincts for beauty; nevertheless, he got it right! Euler begins with the very well known expansion

$$\frac{1}{1-x} = 1 + x + x^2 + x^3 + \cdots + x^n + \cdots . \tag{1}$$

Clearly this expansion is only valid when $|x| < 1$, but that does not stop Euler. Upon putting $x = -1$, he deduces

$$1/2 = 1 - 1 + 1 - 1 + 1 \cdots . \tag{2}$$

To the modern eye, this is a nonsensical statement about divergent series; however following in Euler's bold steps, we won't let that stop us! Indeed, upon applying $x(d/dx)$ to Equation 1 and plugging in $x = -1$, we obtain

$$1/4 = 1 - 2 + 3 - 4 + 5 \cdots . \tag{3}$$

Applying the process again, Euler finds the "trivial zero"

$$0 = 1 - 2^2 + 3^2 - \cdots , \tag{4}$$

and so on. Obviously, Euler is not working with the values at the negative integers of $\zeta(s)$ but rather the function

$$\zeta^*(s) := (1 - 2^{1-s})\zeta(s) = \sum_{n=1}^{\infty} (-1)^{n-1}/n^s , \tag{5}$$

however this is of little consequence and the zeta-values Euler obtains are exactly those rigorously obtained much later by Riemann. (In [Ay1], our $\zeta^*(s)$ is denoted $\phi(s)$.)

Nine years later, in [Eu1] (N.B.: while [Eu1] was published in 1768, it was written in 1749) Euler notices, at least for small $n \geq 2$, that his calculations imply

$$\frac{\zeta^*(1-n)}{\zeta^*(n)} = \begin{cases} \frac{(-1)^{(n/2)+1}(2^n-1)(n-1)!}{(2^{n-1}-1)\pi^n} & \text{if } n \text{ is even} \\ 0 & \text{if } n \text{ is odd.} \end{cases} \tag{6}$$

Upon rewriting Equation 6 using his gamma function $\Gamma(s)$ and the cosine, Euler then "hazards" to conjecture

$$\frac{\zeta^*(1-s)}{\zeta^*(s)} = \frac{-\Gamma(s)(2^s - 1)\cos(\pi s/2)}{(2^{s-1} - 1)\pi^s} , \tag{7}$$

which translates easily into the functional equation of $\zeta(s)$!

Remark 1. Note the important role played by the trivial zeroes in Equation 6 in that they render harmless Euler's inability to calculate explicitly $\zeta^*(n)$, or $\zeta(n)$, at odd integers > 1.

But there is still more! The value $\zeta^*(1)$ is precisely the alternating harmonic series

$$1 - 1/2 + 1/3 - 1/4 \cdots$$

which Euler knows converges to $\log 2$; so his calculations tell him that evaluating the left hand side of Equation 7 at $s = 1$ gives $\frac{1}{2\log 2}$. Euler then takes the limit on the right hand side and obtains the same value! To Euler, this is "strong justification" for his conjecture which Riemann much later proved. (This quote is from Euler's paper [Eu1], the translation is found at the bottom of page 1083 of [Ay1].)

3. A quick introduction to L-series in characteristic p

We now briefly go over the basic definitions of characteristic p L-series. We will present the general definitions but the reader will lose very little by always assuming $A = \mathbb{F}_q[T]$ in what follows.

Let k be an arbitrary global function field of transcendency degree 1 and full field of constants $\mathbb{F}_r$. Let ∞ be a fixed place of k of degree d_∞ over $\mathbb{F}_r$ and let $|?|_\infty$ be the associated absolute value. Let A be the Dedekind domain of those functions regular outside ∞. It is easy to see that the unit group of A is the set of non-zero constants and that one has

$$h_A = d_\infty \cdot h_k , \tag{8}$$

where d_∞ is the degree of ∞ and $h_?$ is the respective class number.

We let K be the completion of k at ∞ and $\mathbb{F}_\infty \simeq \mathbb{F}_{r^{d_\infty}} \subset K$ be the associated finite field. We let $\pi \in K$ be a uniformizing element so that every non-zero element α of K may be written

$$\alpha = \zeta_\alpha \cdot \pi^{n_\alpha} \cdot \langle \alpha \rangle \tag{9}$$

where $\zeta_\alpha \in \mathbb{F}_\infty^*$, $n_\alpha \in \mathbb{Z}$ and $\langle \alpha \rangle \in U_1(K) = \{x \in K \mid |x|_\infty = 1\}$ has absolute value 1. The elements ζ_α and $\langle \alpha \rangle$ depend on our choice of π. The element ζ_α is called the "sign of α" and denoted $\operatorname{sgn}(\alpha)$.

Example 1. When $k = \mathbb{F}_r(T)$ and $A = \mathbb{F}_r[T]$, the simplest choice is $\pi = 1/T$ so that for $n \in A$ monic of degree d, one has

$$n = \pi^{-d}\langle n \rangle = T^d \langle n \rangle \,, \tag{10}$$

with $\langle n \rangle = n/T^d \equiv 1 \pmod{\pi}$.

In general, an element $\alpha \in K^*$ is said to be *monic* or *positive* if and only if $\operatorname{sgn}(\alpha) = \zeta_\alpha = 1$, which clearly depends on the choice of π. Notice that the positive elements clearly form a subgroup of K^*.

Let X be the smooth projective curve associated to k. For any fractional ideal I of A, we let $\deg_k(I)$ be the degree over $\mathbb{F}_r$ of the divisor associated to I on the affine curve $X - \infty$. For $\alpha \in k^*$, one sets $\deg_k(\alpha) = \deg_k((\alpha))$ where (α) is the associated fractional ideal; this clearly gives the correct degree of a polynomial in $\mathbb{F}_r[T]$.

Let $\mathbb{C}_\infty$ be the completion of a fixed algebraic closure $\bar{K}$ of K equipped with the canonical extension of the normalized absolute value on K.

Definition 1. Set $S_\infty := \mathbb{C}_\infty^* \times \mathbb{Z}_p$.

The space S_∞ plays the role of the complex numbers in our theory in that it is the domain of "n^s." Indeed, let $s = (x, y) \in S_\infty$ and let $\alpha \in k$ be positive. The element $u = \langle \alpha \rangle - 1$ has absolute value < 1; thus $\langle \alpha \rangle^y = (1 + u)^y$ is easily defined and computed via the binomial theorem.

Definition 2. We set

$$\alpha^s := x^{\deg_k(\alpha)} \langle \alpha \rangle^y \,. \tag{11}$$

Clearly S_∞ is a group whose operation is written additively. Suppose that $j \in \mathbb{Z}$ and α^j is defined in the usual sense of the canonical $\mathbb{Z}$-action on the multiplicative group. Let $\pi_* \in \mathbb{C}_\infty^*$ be a fixed d_∞-th root of π. Set $s_j := (\pi_*^{-j}, j) \in S_\infty$. One checks easily that Definition 2 gives $\alpha^{s_j} = \alpha^j$. When there is no chance of confusion, we denote s_j simply by "j."

In the basic case $A = \mathbb{F}_r[T]$ one can now proceed to define L-series in complete generality. However, in general there are non-principal ideals. Fortunately there is a canonical and simple procedure to extend Definition 2 to them as follows. Let $\mathcal{I}$ be the group of fractional ideals of the Dedekind domain A and let $\mathcal{P} \subseteq \mathcal{I}$ be the subgroup of principal ideals. Let $\mathcal{P}^+ \subseteq \mathcal{P}$ be the subgroup of principal ideals which have positive generators. It is a standard fact that $\mathcal{I}/\mathcal{P}^+$ is a finite abelian group. The association

$$\mathfrak{h} \in \mathcal{P}^+ \mapsto \langle \mathfrak{h} \rangle := \langle \lambda \rangle \,, \tag{12}$$

where λ is the unique positive generator of $\mathfrak{h}$, is obviously a homomorphism from $\mathcal{P}^+$ to $U_1(K) \subset \mathbb{C}_\infty^*$.

Let $U_1(\mathbb{C}_\infty) \subset \mathbb{C}_\infty^*$ be the group of 1-units defined in the obvious fashion. The binomial theorem again shows that $U_1(\mathbb{C}_\infty)$ is a $\mathbb{Z}_p$-module. However, it is also closed under the unique operation of taking p-th roots; as such $U_1(\mathbb{C}_\infty)$ is a $\mathbb{Q}_p$-vector space.

Lemma 1. *The mapping $\mathcal{P}^+ \to U_1(\mathbb{C}_\infty)$ given by $\mathfrak{h} \mapsto \langle \mathfrak{h} \rangle$ has a unique extension to $\mathfrak{I}$ (which we also denote by $\langle ? \rangle$).*

Proof. As $U_1(\mathbb{C}_\infty)$ is a $\mathbb{Q}_p$-vector space, it is a divisible group; thus the extension follows by general theory. The uniqueness then follows by the finitude of $\mathcal{I}/\mathcal{P}^+$. $\square$

If $s \in S_\infty$ and I as above, we now set

$$I^s := x^{\deg_k(I)} \langle I \rangle^y \,. \tag{13}$$

Thus if $\alpha \in k$ is positive one sees that $(\alpha)^s$ agrees with α^s as in Equation 11.

3.1. **Definition of L-series.** Let $G := \mathrm{Gal}(k^{\mathrm{sep}}/k)$ be the absolute Galois group of k where k^{sep} is a fixed separable closure of k. Let $\bar{\mathbb{Q}}_p$ be a fixed algebraic closure of $\mathbb{Q}_p$ and let $\chi \colon G \to \bar{\mathbb{Q}}_p^*$ be a homomorphism of Galois type; i.e., χ factors through the Galois group G_1 of a *finite* abelian extension k_1 of k. Obviously the image of χ consists of roots of unity and viewing these as sitting in $\mathbb{C}$ (via some injection) allows us to think of χ as also being complex valued. In particular, for each place w of k (including ∞) one attaches a local factor as follows: 1. The place w is ramified for χ; in which case the factor is simply 1. The place w is unramified; in which case the factor is $(1 - \chi(F_w)t)$ where F_w is the (arithmetic) Frobenius element at w.

Let $R_p \subset \bar{\mathbb{Q}}_p$ be the ring of integers with maximal ideal M_p. We fix an injection of R_p/M_p into $\mathbb{C}_\infty$ and so we now obtain local factors in $\mathbb{C}_\infty[t]$ for which we will use the same notation $(1 - \chi(F_w)t)$ etc.

Remark 2. The reader may be wondering why we simply did not use the obvious reduction $\bar{\chi} \colon G \to (R_p/M_p)^*$ to begin with. The answer is that there are no non-trivial p-power roots of unity in characteristic p and so one is hard pressed to get the local factors right. For instance, in the case G_1 is a p-group, the reduced homomorphism $\bar{\chi}$ is the trivial character. If however, we would simply use the trivial character to obtain local factors we would be off at the ramified primes. Thus is is far better to use the characteristic 0 factors in the above fashion.

Let $s \in S_\infty$ and χ as above.

Definition 3. We set

$$L(\chi, s) := \prod_{\substack{v \in \mathrm{Spec}(A) \\ v \text{ unramified}}} (1 - \chi(F_v) v^{-s})^{-1} \,. \tag{14}$$

As in Section 8.9 of [Go2], it is known that $L(\chi, s)$ converges on a "half-plane" of S_∞ and can be analytically extended to an "entire" function on S_∞. Thus, one can view $L(\chi, s)$ as a continuous 1-parameter, where $y \in \mathbb{Z}_p$ is the parameter, family of entire power series in x^{-1} etc.

While we have only discussed abelian χ here for simplicity of exposition, it is clear how to proceed in the non-abelian case.

3.2. **Special polynomials.** Let j now be a non-negative integer with χ, as above and let $s = (x, y) \in S_\infty$.

Definition 4. We set

$$z_L(\chi, x, -j) := L(\chi, \pi_*^j x, -j) \,. \tag{15}$$

It is known that $L(\chi, x, -j)$ is a polynomial in x^{-1} and all such polynomials are called the *special polynomials* of $L(\chi, s)$. By unraveling the definition of $z_L(\chi, x, -j)$, one sees that the coefficients of this polynomial lie in

$$\mathcal{O} := \mathcal{O}_{\mathbf{V}}[\zeta] \tag{16}$$

where ζ is a primitive n-th root of unity and n is the order of the reduction $\bar{\chi}$ of χ (as a finite character) and $\mathcal{O}_{\mathbf{V}} \subset \mathbb{C}_\infty$ is the ring of integers in the *value field* generated by the elements $\{I^{s_1}\}$ (see Subsection 8.2 of [Go2]). As mentioned at the end of [Go5], elementary estimates imply that the degree (in x^{-1}) of $L(\chi, x, -j)$ grows logarithmically in j.

Remark 3. For L-series associated to "τ-sheaves" etc., the logarithmic growth of the special polynomials is due to Böckle [Boc1]. For arbitrary L-series associated to representations of Galois type (not necessarily abelian) one can use Böckle's results and the fact that the Artin Conjecture is true [Go1] for these functions to deduce logarithmic growth.

3.3. **Trivial zeroes.** The classical, characteristic 0 valued, L-series associated to χ also attaches a local factor to the prime ∞ if it is unramified for χ. In the case that χ is non-principal, one knows that this classical L-series is entire; i.e., is a polynomial in $u = r^{-s}$. These infinite factors are missing in the definition of our L-series and thereby equip them with *trivial zeroes* as we shall explain here.

Let ψ be a Hayes-module (i.e., a sign normalized rank one Drinfeld module) [Hay1], Section 7 of [Go2] associated to a twisting of sgn. The module ψ analytically arises from a rank one lattice generated by an element $\xi \in \mathbb{C}_\infty$; one knows that $\xi^{r^{d_\infty}-1} \in K^*$. The extension $K_1 := K(\xi)/K$ is a totally ramified abelian extension with Galois group g_∞ isomorphic to $\mathbb{F}_\infty^*$ via the action on ξ. This local extension is also obtained by adjoining to K *any* non-trivial division point for ψ.

Let $W \subset \bar{\mathbb{Q}}_p$ be the Witt ring of $\mathbb{F}_\infty$ and let $t\colon g_\infty \to \mathbb{F}_\infty^*$ be the homomorphism given by the action of g_∞ on ξ and let $T_\infty\colon g_\infty \to W^*$ be the composition of t and the *Teichmüller character* of $\mathbb{F}_\infty^*$.

We view T_∞ as being extended in the obvious way to a character of the absolute Galois group G_∞ of K^{sep}/K where $K^{\text{sep}} \subset \mathbb{C}_\infty$ is the separable closure.

Let χ_∞ be the local factor at ∞ associated to χ which we also view as a character on G_∞. Assume that for some non-negative j the character $\chi_\infty \cdot T_\infty^j$ is *unramified.* Then, as in Theorem 8.12.5 of [Go2], a double congruence implies that

$$z_L(\chi, x, -j)/(1 - (\chi_\infty \cdot T_\infty^j)(F_\infty)x^{-d_\infty}) \in \mathcal{O}[x^{-1}]\,, \tag{17}$$

where $\mathcal{O}$ is given in Equation 16. Thus zeroes of $(1 - (\chi_\infty \cdot T_\infty^j)(F_\infty)x^{-d_\infty})$ clearly give rise to zeroes of the original L-series $L(\chi, s)$.

Definition 5. The zeroes of $L(\chi, s)$ arising from the factor $(1 - (\chi_\infty \cdot T_\infty^j)(F_\infty)x^{-d_\infty})$ are called *trivial zeroes for χ at $-j$.*

Remark 4. It is clear how to generalize the above construction of trivial zeroes to arbitrary representations of Galois type. For general L-series arising from Drinfeld modules, t-modules, τ-sheaves etc., one proceeds cohomologically as in [Boc1].

Remark 5. If $\chi_\infty \cdot T_\infty^j$ is ramified, then the local factor is 1 and so no non-trivial information is deduced. In this case, it is reasonable to expect that $z_L(\chi, x, -j)$ has no zeroes of absolute value 1.

Definition 6. Let t be a trivial zero for $L(\chi, s)$ at $-j$; so that $\pi_*^j t$ is a root of

$$(1 - (\chi_\infty \cdot T_\infty^j)(F_\infty)x^{-d_\infty})$$

of order $v_0(t)$. Let $v_1(t)$ be the order of t as a zero of $z_L(\chi, x, -j)$. By (17) we know that $v_0(t) \leq v_1(t)$. If this inequality is strict, then we say that t is *non-classical.* The set of all non-negative j such that $L(\chi, s)$ has a non-classical trivial zero at $-j$ will be called *the non-classical set for* $L(\chi, s)$.

Let $\bar{\chi}$ be the reduction of χ considered as a homomorphism into $\mathbb{C}_\infty^*$ via our fixed embedding of R_p/M_p into $\mathbb{C}_\infty$. Let $\mathbb{F}_\chi$ be the finite field generated by the values of $\bar{\chi}$ over the base field $\mathbb{F}_p$; obviously $\mathbb{F}_\chi$ is finite and we let $q_\chi := p^{e(\chi)}$ be its order.

The next proposition is implicit in [Th2] and [DV2]

Proposition 1. *The non-classical set for* $L(\chi, s)$ *is closed under multiplication by* q_χ.

Proof. This follow upon applying the q_χ-th power mapping to the coefficients. □

3.4. **v-adic theory and v-adic trivial zeroes.** Let v be a finite prime of A of degree d_v over $\mathbb{F}_r$. Let k_v be the local field at v with fixed algebraic closure $\bar{k}_v$, equipped with the canonical topology, and let $\mathbb{C}_v$ be the associated complete field. As before let $\mathbf{V}$ be the value field and n the order of the reduction of χ. Fix an embedding $\sigma\colon \mathbf{V}[\zeta] \to \mathbb{C}_v$ where ζ is a primitive n-th root.

Via σ, the functions $\{z_L(\chi, x, -j)\}_{j=0}^\infty$ can be thought of as lying in $\mathbb{C}_v[x^{-d_\infty}]$. Upon setting $x = x_\sigma \in \mathbb{C}_v^*$, they interpolate to a continuous function $L_{\sigma,v}(\chi, x_\sigma, s_\sigma)$ where $(x_\sigma, s_\sigma) \in \mathbb{C}_v^* \times S_{\sigma,v}$; here $S_{\sigma,v} = \mathbb{Z}_p \times \mathbb{Z}/(r^t - 1) = \varprojlim \mathbb{Z}/p^n(r^t - 1)$ is the inductive limit over n of $\mathbb{Z}/p^n(r^t - 1)$ and where $r^t - 1$ is the number of roots of unity in the extension of k_v generated by the image of σ.

As in the previous subsection, let j be chosen so that $\chi_\infty \cdot T_\infty^j$ is unramified at ∞. The local factor $(1 - (\chi_\infty \cdot T_\infty^j)(F_\infty)x^{-d_\infty})$ obviously has roots of unity for its zeroes. Thus, these roots have bounded v-adic absolute value (as of course their absolute value is 1). Their effect on the Newton Polygons for $L_{\sigma,v}(\chi, x_\sigma, s_\sigma)$ is thus very limited and so can essentially be ignored.

However, the process of interpolating v-adically precisely *kills* the Euler factor at v in the following manner. Let σ be extended to the natural action on polynomials via its action on the coefficients.

Proposition 2. *One has*

$$L_{\sigma,v}(\chi, x_\sigma, -j) = \sigma\left((1 - \chi(F_v)v^j x_\sigma^{-d_v}) z_L(\chi, x_\sigma, -j)\right) . \tag{18}$$

Proof. This follows immediately as $\lim\limits_{i\to\infty} v^i = 0$ in $\mathbb{C}_v$. □

We are thus led to a very interesting class of zeroes for $L_{\sigma,v}(\chi, x_\sigma, s_\sigma)$.

Definition 7. The zeroes of $1 - \sigma(\chi(F_v)v^j)x_\sigma^{-d_v}$ in $\mathbb{C}_v$ are called the *v-adic trivial zeroes of* $L_{\sigma,v}(\chi, x_\sigma, s_\sigma)$ *at* $-j \in S_{\sigma,v}$.

The v-adic trivial zeroes are remarkably similar to their counterparts in S_∞. The definition of *non-classical v-adic trivial zeroes* is now obvious as is the definition of the *non-classical set for* $L_{\sigma,v}(\chi, x_\sigma, s_\sigma)$.

Remark 6. Actually, our whole construction of v-adic trivial zeroes *is* non-classical; indeed we know of no analog of our construction of v-adic trivial zeroes in the theory of p-adic L-series. However, we will continue to use "non-classical" to refer to those trivial zeroes whose order is higher than expected.

The next result is the obvious analog of Proposition 1 and has the same proof.

Proposition 3. *The non-classical set for $L_{\sigma,v}(\chi, x_\sigma, s_\sigma)$ is closed under multiplication by q_χ.*

4. The calculations of Thakur and Diaz-Vargas

Let $\chi = \chi_0$ be the trivial character with constant value 1.

Definition 8. We call the function $L(\chi, s)$ the *zeta function of* A and denote it $\zeta_A(s)$. The v-adic interpolations of $\zeta_A(s)$ are denoted $\zeta_{\sigma,v}(x_\sigma, s_\sigma)$.

Clearly one has $\zeta_A(s) = \sum_I I^{-s}$ where I runs over the ideals of A.

The trivial zeroes of $\zeta_A(s)$ then occur at the negative integers $-j \in S_\infty$ where $j \equiv 0 \pmod{r^{d_\infty} - 1}$; indeed the local factor at ∞ in this case is simply $1 - x^{-d_\infty}$. In the case $A = \mathbb{F}_r[T]$, one can show that these zeroes are simple; thus the non-classical set for $\zeta_{\mathbb{F}_r[T]}(s)$ is empty.

Remark 7. Let $A = \mathbb{F}_r[T]$ and let $s = (x, y) \in S_\infty$. In [Wa1], [Sh1] it is shown that for fixed y, a zero of $\zeta_A(x, y)$, has multiplicity 1 and is *uniquely* determined by its absolute value; thus *all* zeroes are simple and must lie in K. Suppose now that θ is a classical Dirichlet character with classical (complex) L-series, $L(\theta, s)$. Let

$$l(\theta, t) := L(\theta, 1/2 + it)\,.$$

At the end of [Go4], it is shown how the classical functional equation combined with the action of complex conjugation imply that the expansion about $t = 0$ of $l(\theta, t)$ is, up to possible multiplication by a non-trivial constant, a *real* power series. Of course the Riemann hypothesis is equivalent to having the zeroes of $l(\theta, t)$ be real; so that classical theory looks quite similar to what was established for $\zeta_A(s)$.

Now let $r = p = 2$.

Example 2. Let $A := \mathbb{F}_2[T_1, T_2]/(T_1^2 + T_1 + T_2^3 + T_2 + 1)$. In this case the quotient field k has genus 1.

Example 3. Let $A := \mathbb{F}_2[T_1, T_2]/(T_1^2 + T_1 + T_2^5 + T_2^3 + 1)$. Here the quotient field k has genus 2.

In both cases, one finds that A has class number 1 which implies that $d_\infty = 1$ also. In both cases, $\zeta_A(s)$ will have trivial zeroes at all negative integers (as $r - 1 = 1$).

Let j be an integer and let $l_p(j)$ be the sum of its p-adic digits. After some hand and machine calculations, Thakur [Th2] established the following result.

Theorem 1. *Let A be as in Example 2 or Example 3. Then the order of vanishing of $\zeta_A(s)$ at $s = -j$ is 2 if $l_2(j) \leq g$ where g is the genus of the quotient field k.*

Thus, in particular, the non-classical set for $\zeta_A(s)$ is non-empty.

For very small j, Thakur also shows the converse to his result. Thus, for instance, in the case of Example 3, the trivial zero at -7 is simple. His paper contains other such examples.

In Theorem 5.4.9 of [Th3], Thakur establishes a partial converse to Theorem 1 in the case of Example 2. More precisely he shows in this case that the trivial zero at $s = -j$ is simple if $l_2(j) = 2$ or $j \equiv 0, 3, 5$ or $6 \pmod 7$.

In [DV2], Javier Diaz-Vargas extends these calculations to more general A where the degree of ∞ is 1 but where A has non-trivial class number and so our exponentiation of non-trivial ideals is used. The same general phenomenon appears to hold.

5. A general conjecture

Let w be any place of k, where k is now completely general, and consider the non-classical set N_w for the interpolation of $L(\chi, s)$ at w (so that if $w = \infty$, this interpolation is $L(\chi, s)$ itself). We know from Propositions 1 and 3 that N_w is closed under multiplication by q_χ. Extrapolating vastly from the calculations presented in the previous section we are led to the following conjecture.

Conjecture 1. The non-classical set N_w consists of elements with *bounded* sum of p-adic digits.

Let $C \geq 0$ be some bound. Then, of course, $\{j \mid l_p(j) \leq C\}$ is a particularly simple set of positive integers which is closed under multiplication by any power of p.

Suppose that one can find infinitely many $-j$ so that the factor $1 - (\chi_\infty \cdot T_\infty^j)(F_\infty)x^{-d_\infty}$ has many zeroes of the same absolute value (which will clearly happen if $d_\infty > 1$). Then, in [Go4], we showed how to construct elements $\alpha \in \mathbb{Z}_p$ so that the power series arising from $L(\chi, x, \alpha)$ had the strange property that there were infinitely many slopes (of the associated Newton Polygon) of length greater than 1. One does this inductively by building up α via its p-adic digits; so that in particular α is built inductively of integers $\{\alpha_i\}$ with $l_p(\alpha_i)$ increasing.

If Conjecture 1 is true, then such counter-examples **cannot** be created out of non-classical trivial zeroes alone. This is illustrated by the next example.

Example 4. Let A be as in Example 2 or 3 and suppose that Conjecture 1 is true in the sense that Thakur's result Theorem 1 is both necessary and sufficient. Then one cannot use the construction in [Go4] to obtain strange α as above. Indeed, once our approximations have sufficiently large sum of p-adic digits the effect of the non-classical trivial zeroes is negligible. In fact, let α_i be an approximation to α with $l_p(\alpha_i) > 2$. Then the trivial zero at $-\alpha_i$ must now be simple and so *cannot* contribute a slope to α of length bigger than 1.

We view Example 4 as being some "justification" for our conjecture in the Eulerian sense of Section 2.

Conjecture 1 is very general but clearly not the final word. One would like to know the exact structure of the non-classical sets as well as the exact orders of the associated trivial zeroes. Moreover, one would like to know how the bounds change as the place w varies, etc. Still Conjecture 1 is a precise statement that indicates a much deeper theory of the zeroes.

Finally, in this paper we have worked with representations of Galois type. It is reasonable to ask for a similar conjecture for arbitrary motives etc. As of this writing, we do not know how to formulate such a conjecture. However, the following simple example indicates some possible structure.

Example 5. Let A be as in Example 2 or 3 so that $p = 2$. Let ψ be the Hayes module associated to A and let $L(\psi, s)$ be its L-series. Then $L(\psi, s) = \zeta_A(s-1)$. Thus j is non-classical for $L(\psi, s)$ if and only if $j+1$ is non-classical for $\zeta_A(s)$. Thus Conjecture 1 implies that $l_p(j+1)$ is bounded. Note that clearly $l_p(j+1)$ can be bounded while $l_p(j)$ goes to infinity (e.g., $j = 2^t - 1, t = 1, 2, \ldots$).

It might be that having $l_p(j+1)$ be bounded instead of $l_p(j)$ is somehow analogous to having a functional equation of the form $s \mapsto k - s$ classically for $k \neq 1$.

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